Dot products

Recall $||\vec{PQ}||$ = distance from P to Q

$$= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Sqrts are hard, linear fcns are easier

Fact $||v||^2 = v \cdot v$

↑ dot product

can use dot prod to understand distance

Dot product is bilinear — linear in each vector

ie. $(v_1 + v_2) \cdot w = v_1 \cdot w + v_2 \cdot w$

vectors

If k is a real # then:

$(kv) \cdot w = k(v \cdot w)$.

Similarly, $v \cdot (w_1 + w_2) = v \cdot w_1 + v \cdot w_2$

$v \cdot (kw) = k(v \cdot w) = (kv) \cdot w$

eg. $||v+w||^2 = (v+w) \cdot (v+w)$

$= v \cdot (v+w) + w \cdot (v+w)$

$= v \cdot v + v \cdot w + w \cdot v + w \cdot w$

$= ||v||^2 + ||w||^2 + 2v \cdot w$

For $\vec{v} = (v_1, v_2, v_3)$ and $\vec{w} = (w_1, w_2, w_3)$

vector; $\underbrace{v_1, v_2, v_3}_{\text{coords (scalars)}}$

then

$\vec{v} \cdot \vec{w} := v_1 w_1 + v_2 w_2 + v_3 w_3$

(dot product in 3-dimensions)

In n dimensions

$\vec{v} = (v_1, \dots, v_n)$

$\vec{w} = (w_1, \dots, w_n)$

then

$\vec{v} \cdot \vec{w} := v_1 w_1 + v_2 w_2 + \cdots + v_n w_n$

$= \sum_{i=1}^{n} v_i w_i$

Basic Important Facts

- given vector $\vec{v}$ and coord vector $\vec{c}$
  then $\vec{v} \cdot \vec{c} = \vec{c} \cdot \vec{v}$ = that coordinate of $\vec{v}$

  eg. $\vec{v} \cdot \vec{i} = \vec{i}$-coord, aka x-coord of $\vec{v}$

  $\vec{v} \cdot \vec{k}$ = z-coord

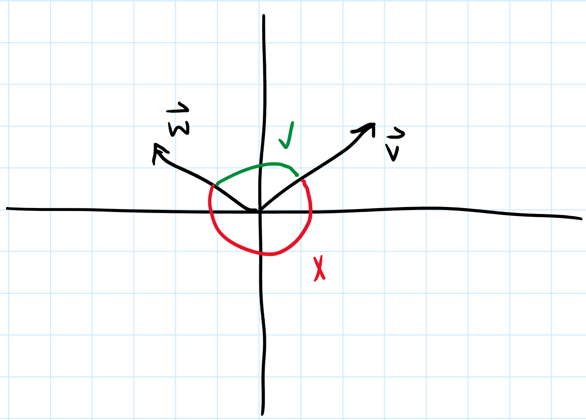
- $||\vec{v}||^2 = \vec{v} \cdot \vec{v}$
  (aka $||\vec{v}|| = \sqrt{\vec{v} \cdot \vec{v}}$)

- bilinearity:
  say $a, b \in \mathbb{R}$
  and $\vec{v}_1, \vec{v}_2, \vec{w}_1, \vec{w}_2$ are vectors

  then
  $(a\vec{v}_1 + b\vec{v}_2) \cdot (\vec{w}_1)$
  $= a(\vec{v}_1 \cdot \vec{w}_1) + b(\vec{v}_2 \cdot \vec{w}_1)$
  and
  $\vec{v}_1 \cdot (a\vec{w}_1 + b\vec{w}_2)$
  $= a(\vec{v}_1 \cdot \vec{w}_1) + b(\vec{v}_1 \cdot w_2)$

Angles

given $\vec{v}, \vec{w}$ "angle" is the smallest angle btwn them



Fact

$\vec{v} \cdot \vec{w} = ||\vec{v}|| \, ||\vec{w}|| \cos\theta$

θ = angle btwn them

$$\Rightarrow \cos\theta = \frac{\vec{v}\cdot\vec{w}}{\|\vec{v}\|\,\|\vec{w}\|}$$

Note: $0 \leq \theta \leq \pi$
$0^\circ \leq \theta \leq 180^\circ$

In particular, $\vec{v}\cdot\vec{w}=0$ iff $\vec{v}, \vec{w}$ are perpendicular

- Notice that if $\vec{u}$ is any vector, then $\vec{u}\cdot\vec{u} \geq 0$
"Trivial Inequality"
eg $\vec{u} = \vec{v} - \vec{w}$

So

$$\vec{u}\cdot\vec{u} = (\vec{v}-\vec{w})\cdot(\vec{v}-\vec{w})$$
$$= \vec{v}\cdot\vec{v} + \vec{w}\cdot\vec{w} - 2\cdot\vec{v}\cdot\vec{w} \geq 0$$

bilinearity

$$\Rightarrow \vec{v}\cdot\vec{w} \leq \frac{\vec{v}\cdot\vec{v} + \vec{w}\cdot\vec{w}}{2}$$

Note expanding $\|\vec{v}+\vec{w}\|^2$ using bilinearity gives law of cosines

Cauchy-Schwarz Inequality

$$|\vec{v}\cdot\vec{w}| \leq \sqrt{\|\vec{v}\|^2\cdot\|\vec{w}\|^2} = \|\vec{v}\|\cdot\|\vec{w}\|$$

Square both sides:

$$(\vec{v}\cdot\vec{w})(\vec{v}\cdot\vec{w}) = (\vec{v}\cdot\vec{w})^2 \leq (\vec{v}\cdot\vec{v})(\vec{w}\cdot\vec{w})$$

equivalent to: $|\cos\theta| \leq 1$

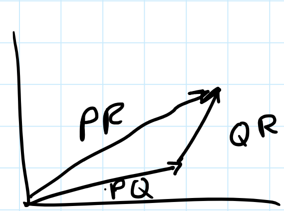
Triangle Inequality

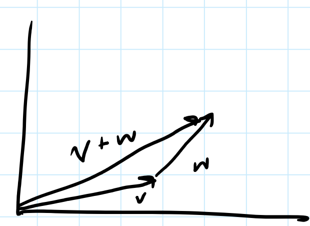
Three equivalent Forms

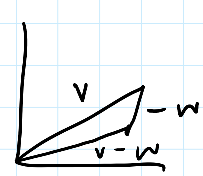
① $\|\vec{PR}\| \leq \|\vec{PQ}\| + \|\vec{QR}\|$

② $\|\vec{v}+\vec{w}\| \leq \|\vec{v}\| + \|\vec{w}\|$

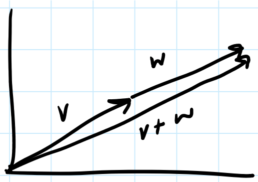
③ $\|\vec{v}\| - \|\vec{w}\| \leq \|\vec{v}-\vec{w}\|$

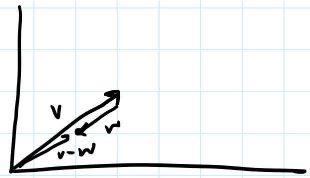






Note equality in ② and ③ iff $\vec{v}$ and $\vec{w}$ in same direction





<u>Note</u> ② is equivalent to

$$(\vec{v}+\vec{w})\cdot(\vec{v}+\vec{w}) = \|\vec{v}+\vec{w}\|^2 \le (\|\vec{v}\| + \|\vec{w}\|)^2$$

use bilinearity on the left, this is just

$$\|\vec{v}\|^2 + \|\vec{w}\|^2 + 2\vec{v}\cdot\vec{w} \le \|\vec{v}\|^2 + \|\vec{w}\|^2 + 2\|\vec{v}\|\|\vec{w}\|$$

equivalent to Cauchy-Schwarz

<u>Cross Product</u>

$\vec{v} = (v_1, v_2, v_3) \qquad \vec{w} = (w_1, w_2, w_3)$

$$\vec{v}\times\vec{w} = (v_2w_3 - v_3w_2, -v_1w_3 + v_3w_1, v_1w_2 - v_2w_1)$$

$$\begin{array}{cccc} \vec{v} & v_1 & v_2 & v_3 \\ \vec{w} & w_1 & w_2 & w_3 \end{array} \rightarrow \left(\begin{array}{c} \cancel{v_1} \\ \cancel{w_1} \end{array} \underset{①}{\begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix}}, \underset{②}{\begin{vmatrix} v_1 & \cancel{v_2} & v_3 \\ w_1 & \cancel{w_2} & w_3 \end{vmatrix}}, \underset{③}{\begin{vmatrix} v_1 & v_2 & \cancel{v_3} \\ w_1 & w_2 & \cancel{w_3} \end{vmatrix}} \right)$$

$$\vec{v}\times\vec{w} : \left(\underbrace{\begin{vmatrix} v_2 & v_3 \\ w_2 & w_2 \end{vmatrix}}_{\text{Determinant}}, \begin{vmatrix} v_3 & v_1 \\ w_3 & w_1 \end{vmatrix}, \begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix} \right)$$

same as:

$$-\begin{vmatrix} v_1 & v_3 \\ w_1 & w_3 \end{vmatrix}$$

<u>"Anti-Symmetric"</u>

$\vec{v}\times\vec{w} = -\vec{w}\times\vec{v}$

<u>Determinants</u>

say $\vec{u} = (u_1, u_2, u_3)$

$$\begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

cofactor expansion (expansion by minors)

$$= u_1 \begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix} - u_2 \begin{vmatrix} v_1 & v_3 \\ w_1 & w_3 \end{vmatrix} + u_3 \begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix}$$

$$= (u_1, u_2, u_3) \cdot \left(\begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix}, \begin{vmatrix} v_3 & v_1 \\ w_3 & w_1 \end{vmatrix}, \begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix} \right)$$

$$= \vec{u} \cdot (\vec{v} \times \vec{w})$$

↳ combining dot product & cross product gives determinant

$= \pm$ volume of the parallelpiped determined by $\vec{u}, \vec{v}, \vec{w}$

<u>Note</u>: Determinant is always $\emptyset$ if 2 columns are the same

so $\vec{v} \cdot (\vec{v} \times \vec{w}) = 0$

$\Rightarrow$ $\vec{v}$ is perpendicular to $\vec{v} \times \vec{w}$

& $\vec{w} \perp$ to $\vec{v} \times \vec{w}$

<u>Direction</u> of cross product is that it is perpendicular to $\vec{v}$ and $\vec{w}$

(ie perpendicular to the plane spanned by $\vec{v}$ and $\vec{w}$)

perpendicular to v and w
(ie perpendicular to the plane spanned by $\vec{v}$ and $\vec{w}$)

What if $\vec{v}$ & $\vec{w}$ don't span a plane, ie, they are parallel?
Then $\vec{v} \times \vec{w} = \varnothing$

<u>Fact</u> $\|\vec{v} \times \vec{w}\| = \|\vec{v}\| \|\vec{w}\| \sin\theta$

θ = angle btwn them

<u>Remark</u> $\|\vec{v} \times \vec{w}\| = \|\vec{v}\| \|\vec{w}\|$ iff $\vec{v}$ and $\vec{w}$ are $\perp$

<u>Notice</u> θ = angle btwn $\vec{v}, \vec{w}$

$$\cos\theta = \frac{\vec{v} \cdot \vec{w}}{\|\vec{v}\| \cdot \|\vec{w}\|}$$

$$\cos^2\theta = \frac{(\vec{v} \cdot \vec{w})^2}{(\vec{v} \cdot \vec{v})(\vec{w} \cdot \vec{w})}$$

$$\sin\theta = \frac{\|\vec{v} \times \vec{w}\|}{\|\vec{v}\| \cdot \|\vec{w}\|}$$

$$\sin^2\theta = \frac{(\vec{v} \times \vec{w}) \cdot (\vec{v} \times \vec{w})}{(\vec{v} \cdot \vec{v})(\vec{w} \cdot \vec{w})}$$

$$\Rightarrow \frac{(\vec{v} \cdot \vec{w})^2 + (\vec{v} \times \vec{w}) \cdot (\vec{v} \times \vec{w})}{(\vec{v} \cdot \vec{v})(\vec{w} \cdot \vec{w})} = 1$$

$$\Leftrightarrow \underbrace{(\vec{v} \cdot \vec{w})^2 + (\vec{v} \times \vec{w}) \cdot (\vec{v} \times \vec{w}) = (\vec{v} \cdot \vec{v})(\vec{w} \cdot \vec{w})}$$

gives error term of Cauchy Swarz

because C-S says:
$(\vec{v} \cdot \vec{w})^2 \leq (\vec{v} \cdot \vec{v})(\vec{w} \cdot \vec{w})$
and now we know that
the difference/error $= (\vec{v} \times \vec{w}) \cdot (\vec{v} \times \vec{w})$

<u>Note</u> $\|\vec{v} \times \vec{w}\|$ is the area of the parallelogram determined by $\vec{v}$ and $\vec{w}$

Half of it is the area of the triangle

Half of it is the area of the triangle

$\vec{v} \times \vec{w}$ is <u>bilinear</u> in $\vec{v}$ and $\vec{w}$

ie. given $a, b \in \mathbb{R}$

$\vec{v}_1, \vec{w}_1, \vec{v}_2, \vec{w}_2$ vectors

$(a\vec{v}_1 + b\vec{v}_2) \times \vec{w}_1$

$= a(\vec{v}_1 \times \vec{w}_1) + b(\vec{v}_2 \times \vec{w}_1)$

and similarly ....

symmetric

⚠ Recall $\vec{v} \cdot \vec{w} = \vec{w} \cdot \vec{v}$

$\vec{v} \times \vec{w} = -\vec{w} \times \vec{v}$

anti-symmetric

Consider

$\vec{r} = \vec{u} \times (\vec{v} \times \vec{w})$

<u>Recall</u> $\vec{v} \times \vec{w} \perp \vec{v}$

$\vec{v} \times \vec{w} \perp \vec{w}$

$\vec{r} \perp \vec{u}$

$\vec{r} \perp \vec{v} \times \vec{w}$

As long as $\vec{v} \times \vec{w} \neq 0$, $\vec{v} \times \vec{w} \perp$ to plane spanned by $\vec{v}$ and $\vec{w}$

Also, the plane spanned by $\vec{v}, \vec{w}$ is set of vectors that are $\perp$ to $\vec{v} \times \vec{w}$

$\Rightarrow \vec{r}$ is in the plane spanned by $\vec{v}$ and $\vec{w}$

$\Rightarrow \vec{r}$ is $a\vec{v} + b\vec{w}$ for $a, b \in \mathbb{R}$

$a = \vec{u} \cdot \vec{w}$

$b = -\vec{u} \cdot \vec{v}$